Sydney Technical High School

Name: ....

Maths Class: ....

Year 12
Mathematics Extension 2

HSC Course
Trial Examination
August, 2022

Time allowed: 3 hours + 10 minutes reading time

General Instructions:

- Marks for each question are indicated on the question.
- NESA approved calculators may be used
- In Questions 11-16, show all relevant mathematical reasoning and/or calculations
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black pen
- All answers are to be in the writing booklet provided
- A reference sheet is provided.

Total marks – 70

Section I - 10 Marks
- Attempt Question 1-10 on the sheet provided
- Allow about 15 minutes for this section

Section II – 90 Marks
- Attempt Questions 11-14
- Allow about 2 hours and 45 minutes for this section

Section I

10 Marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1-10.

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1. Consider the statement:

$$\forall x: |2x+5| \le 9 \Rightarrow |x| \le 4$$

Which of the following is a counterexample to this statement?

(A) $x = 0$

(B) $x = -6$

(C) $x = -10$

(D) $x = 10$

2. What is the magnitude of the vector $(\cos\theta)\underset{\sim}{i} + (\sin\theta)\underset{\sim}{j} + (\tan\theta)\underset{\sim}{k}$?

(A) 1

(B) $\operatorname{cosec}\theta$

(C) $\cot\theta$

(D) $\sec\theta$

3. Where $\lambda \in \mathbb{R}$, which of the following is a vector equation of the line through $(5,3)$, parallel to $2\underset{\sim}{i} + 4\underset{\sim}{j}$?

(A) $2\underset{\sim}{i} + 4\underset{\sim}{j} + \lambda(5\underset{\sim}{i} + 3\underset{\sim}{j})$

(B) $2\underset{\sim}{i} + 4\underset{\sim}{j} + \lambda(3\underset{\sim}{i} - 5\underset{\sim}{j})$

(C) $5\underset{\sim}{i} + 3\underset{\sim}{j} + \lambda(\underset{\sim}{i} + 2\underset{\sim}{j})$

(D) $5\underset{\sim}{i} + 3\underset{\sim}{j} + \lambda(2\underset{\sim}{i} - \underset{\sim}{j})$

4. Given $z = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$, which of the following is $(\bar{z})^{-1}$ in polar form?

(A) $\frac{1}{3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

(B) $-\frac{1}{3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

(C) $\frac{1}{3}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

(D) $-\frac{1}{3}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

5. Which of the following is equivalent to i^i?

(A) $e^{-\frac{\pi}{2}}$

(B) $e^{\frac{\pi}{2}}$

(C) i

(D) $-i$

6. State the values of A, B, and C if

$$\frac{4x+10}{(2-x)(x^2+2)} = \frac{A}{2-x} + \frac{Bx+C}{x^2+2}$$

(A) $A = 3, B = -1, C = 5,$

(B) $A = 3, B = 3, C = 2$

(C) $A = 5, B = -1, C = 0$

(D) $A = 5, B = -6, C = 5$

7. Consider the statement: "If I have power, then I have responsibility."

Which of the following is NOT an equivalent statement?

(A) If I do not have responsibility, then I do not have power.

(B) If I have responsibility, then I have power.

(C) Either I do not have power, or I have responsibility.

(D) I cannot have both power and no responsibility.

8. The velocity of a particle varies with respect to its displacement, such that $v = 9 - 6x - x^2$, what is the acceleration when $x = 0$?

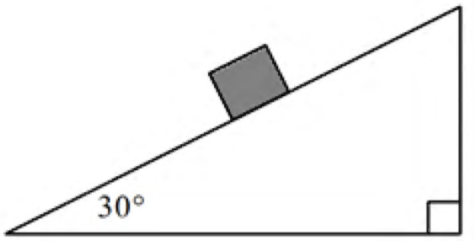
(A) 6

(B) -6

(C) 54

(D) -54

9. An object is placed on a smooth ramp inclined at an angle of 30° to the horizontal.



If the acceleration due to gravity is 10 ms^{-2}, at what velocity is the object travelling down the ramp after t seconds?

(A) $5t\ ms^{-1}$

(B) $5\ ms^{-1}$

(C) $5\sqrt{3}t\ ms^{-1}$

(D) $5\sqrt{3}\ ms^{-1}$

10. Which of the following integrals is greater than zero?

(A)
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{x^2} \cos x\, dx$$

(B)
$$\int_{-\pi}^{\pi} x^3 \cos x\, dx$$

(C)
$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sin^2 x - \cos^2 x) dx$$

(D)
$$\int_{-1}^{1} \sin^{-1}(x^3)\, dx$$

Section II

Total marks – 90
Attempt Question 11-16
Allow about 2 hour and 45 minutes for this section

Begin each question on a NEW page

In Questions 11-14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Begin a NEW page.

a) If $z = 5 + i$ and $w = 4 - 2i$, simplify in $x + iy$ form

i) $z + iw$ **1**

ii) $z\overline{w}$ **1**

iii) $\frac{w}{z}$ **2**

b) For the complex number $z = -3 - 3i$

i) Express z in the form $r(\cos\theta + i\sin\theta)$. **2**

ii) Hence or otherwise, find the smallest positive integer n, such that $z^n \in \mathbb{R}$. **2**

c) Prove that $x^2 + x + 1$ is odd for all positive integer values of x. **2**

d) If $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} + 5\underset{\sim}{j} + 6\underset{\sim}{k}$, evaluate $\left(\underset{\sim}{a} - 2\underset{\sim}{b}\right) \cdot \underset{\sim}{a}$. **1**

e) Find the vector $\underset{\sim}{v}$ parallel to $\underset{\sim}{u} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}$ that has a magnitude of 3. **2**

f) Find: **2**

$$\int \frac{2x + 3}{x + 1}\,dx$$

End of Question 11

Question 12 (15 marks) Begin a NEW page.

a) Where $z = x + iy$, sketch the graph defined by $Im(z^2) = 4$. **2**

b) Find the following integrals

i) **2**

$$\int \frac{1}{\sqrt{5 + 4x - x^2}} dx$$

ii) **3**

$$\int \sin^4 x \cos^3 x \; dx$$

c) Evaluate **2**

$$\int_1^{\sqrt{5}} x^3 \sqrt{x^2 - 1} \, dx$$

d) A particle is moving in a straight line in simple harmonic motion. The amplitude of the motion is 4 cm, and the period of the motion is 8 seconds. The particle begins at the maximum displacement from the centre of motion.

i) Find when the particle is first 2 cm to the right of the centre of motion. **2**

ii) Calculate the maximum velocity of the particle. **2**

iii) Find the maximum acceleration, and the first time at which this occurs. **2**

End of Question 12

Question 13 (15 marks) Begin a NEW page.

a) It is known that $P(x) = 2x^3 - 19x^2 + 112x + d$, where $d \in \mathbb{R}$, has a zero of $5 + 6i$.

i) What are the two other zeroes of $P(x)$? **2**

ii) Find the value of d. **2**

b) If ω is a non-real cube root of unity, prove that:

i) $1 + \omega + \omega^2 = 0$ **1**

ii) $(1 + \omega^2)(1 + \omega - \omega^2)^7 = 128$ **2**

c) For integers a, b, c, with $b, c \leq 9$, the number x can be written as $100a + 10b + c$. **3**
For example, the number 56789 can be written as $100 \times 567 + 10 \times 8 + 9$.

Use this to prove that x is divisible by 4 **if and only if** the last two digits are divisible by 4.

d) Let $\underset{\sim}{a} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} 2 \\ -4 \\ 4 \end{bmatrix}$. If θ is the acute angle between these vectors, find the exact value of $\sin 2\theta$. **3**

e) Find the point of intersection of the vector lines **2**

$$\underset{\sim}{r} = \begin{bmatrix} 0 \\ 3 \\ -5 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \text{ and } \underset{\sim}{q} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

End of Question 13

Question 14 (15 marks) Begin a NEW page.

a) A particle moves with equation of motion $x = \sin 3t + \cos 3t + 2$ metres. Prove that the particle is moving in simple harmonic motion and find the centre and period of its motion. **2**

b) For integers a and b, with $b \neq 0$, consider the statement:

"If x is irrational, then $a + bx$ is irrational."

i) State the contrapositive of this statement. **1**

ii) Hence, or otherwise, show that the statement is true. **2**

iii) Show that $\sqrt{3}$ is irrational, and hence deduce that $5 + 7\sqrt{3}$ is irrational. **2**

c) Determine whether the origin $(0, 0, 0)$ lies inside, outside, or on the sphere given by the equation $x^2 - 2x + y^2 + z^2 + 4z + 4 = 0$. **2**

d) The line L_1 is given by $\underset{\sim}{r} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix}$ and the line L_2 passes the points $(3, 8, -2)$ and $(a, -2, 10)$. Find the value of a if L_1 is parallel to L_2. **2**

e) Where a is a positive constant

i) Show that **2**

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

ii) Hence, or otherwise, evaluate **2**

$$\int_0^1 x(1-x)^{50}\,dx$$

End of Question 14

Question 15 (15 marks) Begin a NEW page.

a) The acceleration of a particle is given by $\ddot{x} = -\frac{1}{2}e^{-4x}$.
Initially, the particle is at the origin with a velocity of $\dot{x} = \frac{1}{2}ms^{-1}$.

i) Express velocity as a function of displacement. **2**

ii) Find the displacement as a function of time. **2**

iii) Describe the limiting behaviour of displacement and velocity (as $t \to \infty$). **1**

b) For the integral

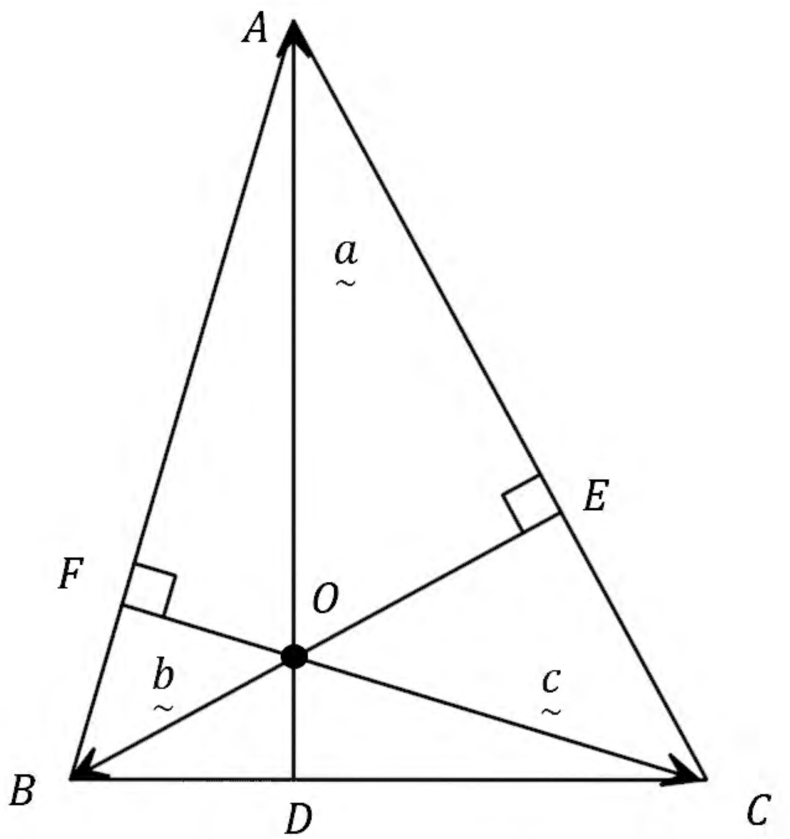
$$I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx$$

i) Show that $I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$ **3**

ii) Hence, find the value of I_4 in terms of π. **2**

Question 15 continues on Page 10

c) In the diagram below, $\underset{\sim}{a} = \overrightarrow{OA}$, $\underset{\sim}{b} = \overrightarrow{OB}$, and $\underset{\sim}{c} = \overrightarrow{OC}$.
ΔABC is an acute angled triangle. O is the point of intersection of altitudes CF and BE (i.e. $BE \perp AC$ and $CF \perp AB$). $\overrightarrow{AO}$ is produced from O to point D on BC.



i) Show that $\underset{\sim}{b} \cdot \left(\underset{\sim}{c} - \underset{\sim}{a}\right) = 0$ and $\underset{\sim}{c} \cdot \left(\underset{\sim}{b} - \underset{\sim}{a}\right) = 0$. **2**

ii) Hence, show that $AD \perp BC$. **2**

iii) What geometrical property of the triangle has been proven? **1**

End of Question 15

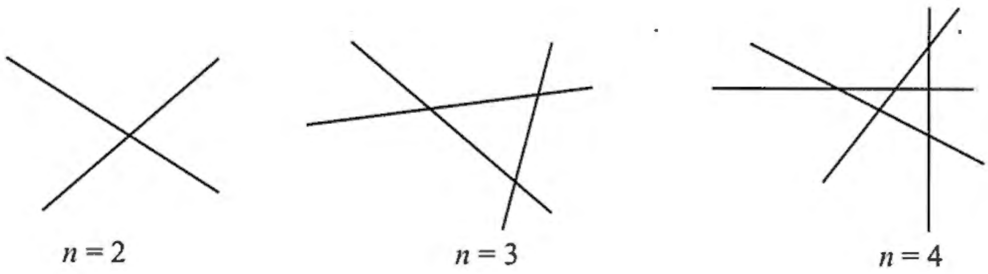
Question 16 (15 marks) Begin a NEW page.

a)

i) Show that $(1 + i\tan\theta)^n + (1 - i\tan\theta)^n = \frac{2\cos n\theta}{\cos^n\theta}$ **2**
Where $\cos\theta \neq 0$ and n is a positive integer.

ii) Hence show that if z is a purely imaginary number, then the roots of **2**
$(1+z)^4 + (1-z)^4 = 0$ are $z = \pm i\tan\frac{\pi}{8}, \pm i\tan\frac{3\pi}{8}$.

b) Consider n lines drawn on a plane, as shown in the series of diagrams below. Let X_n be the maximum number of possible points of intersection of these lines.



i) Find the values of a, b, and c in the table below **1**

n	2	3	4	5	6
X_n	1	3	a	b	c

ii) Prove by mathematical induction that for $n \geq 2$, X_n is given by **3**

$$X_n = \frac{n(n-1)}{2}$$

c) Prove the following, where x, a, b, c are real positive numbers.

i) $x + \frac{1}{x} \geq 2$ **2**

ii) Hence, show $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$ **2**

iii) Hence, show $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$ **3**

End of Examination

 X2 Solutions Trial

1. (B) $|2x-6+5| = 7$ ✓

$|-6| > 4$ ×

2. (D) $\sqrt{\cos^2\theta+\sin^2\theta+\tan^2\theta} = \sqrt{1+\tan^2\theta}$

$= \sec\theta$

3. (C) as $\underset{\sim}{i}+2\underset{\sim}{j}$ is parallel to $2\underset{\sim}{i}+4\underset{\sim}{j}$

4. (A) $\dfrac{1}{3\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)} = \dfrac{1}{3}\,\dfrac{\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}}{\left(\cos^2\frac{\pi}{6}+\sin^2\frac{\pi}{6}\right)} =$

5. (A) $i^i = \left(e^{\frac{\pi}{2}i}\right)^i = e^{-\frac{\pi}{2}}$.

6. (B) $4x+10 = A(x^2+2) + (Bx+C)(2-x)$

At $x=2$ At $x=0$

$18 = A\times 6$ $10 = 3\times 2 + C\times 2$

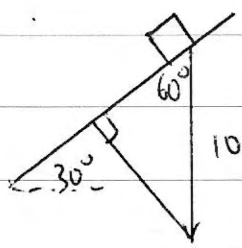
$A=3$ $C=2$

7. (B) Converse is not equivalent.

8. (D) $a = v\times\dfrac{dv}{dx} = (-6-2x)(9-6x-x^2)$

$\therefore$ At $x=0$ $a=-6\times 9 = -54$

9. (A)



$a = 10\cos 60°$

$= 5$

$\therefore v = 5t$.

10. (A) $e^{x^2}\cos x \geq 0$ and even, in $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

11.

a) $z = 5+i, \quad w = 4-2i$

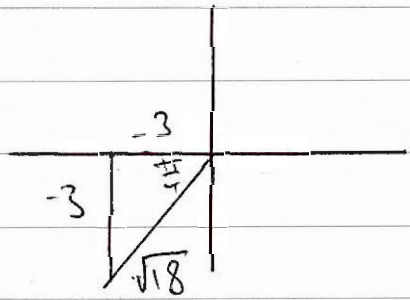
i) $z + iw = 5+i+i(4-2i)$
$= 5+i+4i+2$
$= 7+5i$

ii) $z\bar{w} = (5+i)(4+2i)$
$= 20+10i+4i-2$
$= 18+14i$

iii) $\frac{w}{z} = \frac{4-2i}{5+i} \times \frac{5-i}{5-i}$
$= \frac{20-10i-4i-2}{25+1}$
$= \frac{18}{26} - \frac{14}{26}i = \frac{9}{13} \quad \frac{7}{13}i$

b) $z = -3-3i$

i)



$\therefore z = 3\sqrt{2}\left(\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right)$

ii) $\arg(z^n) = \frac{-3n\pi}{4}$
As $\sin(k\pi) = 0, \ k \in \mathbb{Z}, \ n = 4.$

c) $x^2 + x + 1$

$= x(x+1) + 1$

If x is odd, $x+1$ is even, so

$x(x+1) + 1$ is odd

If x is even

$x(x+1) + 1$ is odd.

d) $(\underset{\sim}{a} - 2\underset{\sim}{b}) \cdot \underset{\sim}{a}$

$= (\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k} - 2(4\underset{\sim}{i} + 5\underset{\sim}{j} + 6\underset{\sim}{k})) \cdot (\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k})$

$= (-7\underset{\sim}{i} - 8\underset{\sim}{j} + 9\underset{\sim}{k}) \cdot (\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k})$

$= -7 - 16 - 27$

$= -50$

e) $|\underset{\sim}{u}| = \sqrt{29}$

$\therefore \underset{\sim}{v} = \frac{3}{\sqrt{29}}(3\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k})$

f) $\int \frac{2x+3}{x+1}\,dx = \int \frac{2x+2+1}{x+1}\,dx$

$= \int 2 + \frac{1}{x+1}\,dx$

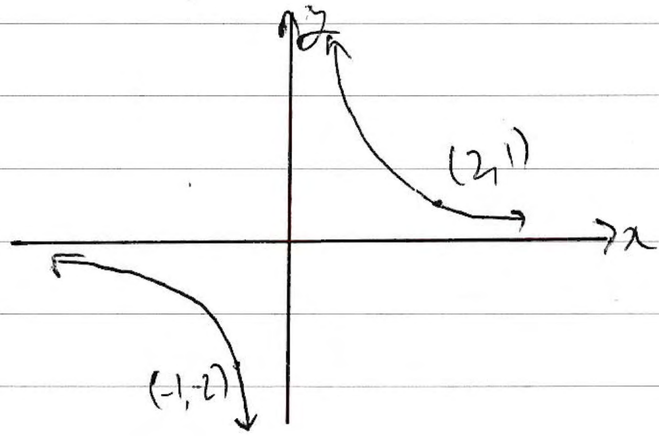
$= 2x + \ln|x+1| + c$

12. a) $\text{Im}(x+iy^2) = 4$

$\text{Im}(x^2-y^2+2xyi) = 4$

$2xy = 4$

$xy = 2$



b) i) $\displaystyle\int \frac{1}{\sqrt{5+4x-x^2}}\,dx$

$\displaystyle = \int \frac{1}{\sqrt{5-(x^2-4x+4-4)}}\,dx$

$\displaystyle = \int \frac{1}{\sqrt{9-(x-2)^2}}\,dx$

$\displaystyle = \sin^{-1}\left(\frac{x-2}{3}\right) + C$

ii) $\int \sin^4 x \cos^3 x \, dx$

$= \int \sin^4 x \cos^3 x \cos x \, dx$ $\quad$ $u = \sin x$, $du = \cos x \, dx$

$= \int u^4 (1-u^2) \, du$

$= \int u^4 - u^6 \, du$

$= \frac{1}{5}u^5 - \frac{1}{7}u^7 + C$

$= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C.$

c) $\int_1^{\sqrt{5}} x^3 \sqrt{x^2-1} \, dx$ $\quad$ $u = x^2 - 1$ $\quad$ $x^2 = u+1$

$x = 1 \rightarrow u = 0$

$= \frac{1}{2}\int_1^{\sqrt{5}} 2x \times x^2 \times \sqrt{x^2-1} \, dx$ $\quad$ $x = \sqrt{5} \rightarrow u = 4$

$du = 2x \, dx$

$= \frac{1}{2}\int_0^4 (u+1)\sqrt{u} \, du$

$= \frac{1}{2}\int_0^4 (u+1) u^{\frac{1}{2}} \, du$

$= \frac{1}{2}\int_0^4 u^{\frac{3}{2}} + u^{\frac{1}{2}} \, du$

$= \frac{1}{2} \cdot \left[\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} \right]_0^4$

$= \frac{1}{2}\left(\frac{2}{5} \times 2^5 + \frac{2}{3} \times 2^3 - 0 \right)$

$= \frac{136}{15}.$

d) $a = 4$ $\quad \frac{2\pi}{n} = 8$ $\quad$ At $t = 0$, $x = 4$.

$n = \frac{\pi}{4}$

$\therefore$ Use $\quad x = 4\cos\left(\frac{\pi}{4}t\right)$

i) When $x = 2$.

$$2 = 4\cos\left(\frac{\pi}{4}t\right)$$

$$\cos\left(\frac{\pi}{4}t\right) = \frac{1}{2}$$

$$\frac{\pi}{4}t = \frac{\pi}{3}$$

$$t = \frac{4}{3}$$

ii) $\dot{x} = -4 \times \frac{\pi}{4}\sin\left(\frac{\pi}{4}t\right)$

$= -\pi\sin\left(\frac{\pi}{4}t\right)$

$\therefore$ Max $\dot{x} = -\pi \times -1$

$= \pi$ cm/s.

iii) $\ddot{x} = -\frac{\pi^2}{4}\cos\left(\frac{\pi}{4}t\right)$

$\therefore$ Max acceleration is $\frac{\pi^2}{4}$, occurs at

$$\cos\left(\frac{\pi}{4}t\right) = -1$$

$$\frac{\pi}{4}t = \pi$$

$$t = 4$$

13. a) i) $P(x) = 2x^3 - 19x^2 + 112x + d.$

Zeroes $\alpha = 5+6i, \quad \beta = 5-6i, \quad \gamma$

$$\alpha + \beta + \gamma = \frac{19}{2}$$

$$5+6i + 5-6i + \gamma = \frac{19}{2}$$

$$\gamma + 10 = \frac{19}{2}$$

$$\gamma = -\frac{1}{2}.$$

iii) $\alpha\beta\gamma = \frac{-d}{2}$

$$-\frac{1}{2}(5-6i)(5+6i) = -\frac{d}{2}$$

$$d = 25 + 36$$

$$= 61$$

b) i) w solves $z^3 - 1 = 0$

$$(z-1)(z^2+z+1) = 0$$

w - non real, so it solves $z^2+z+1=0$

$$\therefore w^2 + w + 1 = 0.$$

ii) From i)

$1 + w^2 = -w$ — (1) $\qquad 1 + w - w^2 = -2w^2$ — (2)

(1) $\times$ (2)^7: $(1+w^2)(1+w-w^2)^7 = -w \times (-2w^2)^7$

$$= -w \times -128w^{14}$$

$$= 128\, w^{5\times 3}$$

$$= 128$$

c) $\Rightarrow$: If x is divisible by 4, $x = 4M$, $M \in \mathbb{Z}$.

$$\therefore 100a + 10b + c = 4M$$
$$\begin{aligned} 10b + c &= 4M - 100a \\ &= 4(M - 25) \\ &= 4L, \quad L \in \mathbb{Z}. \end{aligned}$$

$\therefore$ $10b+c$ is divisible by 4, with $10b+c$ as the last two digits.

$\Leftarrow$: If the last two digits are divisible by 4, $10b+c = 4k$, $k \in \mathbb{Z}$.

$$\begin{aligned} \therefore x &= 100a + 10b + c \\ &= 100a + 4k \\ &= 4(25a + k) \\ &= 4m, \quad m \in \mathbb{Z}. \end{aligned}$$

$\therefore$ x is divisible by 4.

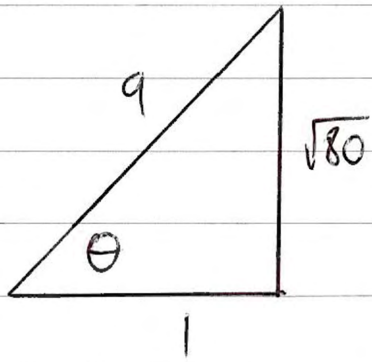
$\therefore$ x is divisible by 4 $\Leftrightarrow$ The last two digits are divisible by 4.

d) $\underset{\sim}{a}\cdot\underset{\sim}{b} = 2-8+8$ $\qquad |\underset{\sim}{a}| = \sqrt{9}$ $\qquad |\underset{\sim}{b}| = \sqrt{36}$

$= 2$ $\qquad = 3$ $\qquad = 6$

$\therefore 3\times 6 \times \cos\theta = 2$

$\cos\theta = \frac{1}{9}$



$$\therefore \sin 2\theta = 2\sin\theta\cos\theta$$

$$= 2\times\frac{\sqrt{80}}{9}\times\frac{1}{9}$$

$$= \frac{8\sqrt{5}}{81}$$

e) If $\begin{bmatrix} 0 \\ 3 \\ -5 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} -\lambda \\ \lambda \\ 2\lambda \end{bmatrix} = \begin{bmatrix} -\mu - 1 \\ 2\mu - 1 \\ -\mu + 8 \end{bmatrix}$$

$\therefore -\lambda = -\mu - 1 \quad —(1)$

$\lambda = 2\mu - 1 \quad —(2)$

$2\lambda = -\mu + 8 \quad —(3)$

$(1) + (2): \quad 0 = \mu - 2 \quad —(3)$

$(3) \rightarrow (2) \qquad 2 \times 2 - 1 = \lambda$

$\lambda = 3.$

Test in (3)

$LHS = 6 \qquad RHS = -2 + 8$

$= 6$ ✓

$\therefore$ Point at $\lambda = 3$: $\begin{bmatrix} 0 \\ 3 \\ -5 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} -3 \\ 6 \\ 1 \end{bmatrix}$$

$\therefore$ Point $(-3, 6, 1)$

14

a) $x = \sin 3t + \cos 3t + 2$

$\dot{x} = 3\cos 3t - 3\sin 3t$

$\ddot{x} = -9\sin 3t - 9\cos 3t$

$= -9(\sin 3t + \cos 3t + 2 - 2)$

$= -9(x-2)$

$\therefore$ SHM with centre of motion $x=2$ and Period $\frac{2\pi}{3}$.

b) i) If $a+bx$ is rational, x is rational.

ii) If $a+bx$ is rational:

$a+bx = \frac{p}{q} \quad p, q \in \mathbb{Z},\ q \neq 0.$

$bx = \frac{p}{q} - a$

$x = \frac{p-qa}{qb} \quad p-qa \in \mathbb{Z} \quad qb \in \mathbb{Z}$

$\therefore x$ is rational.

By contrapositive, if x is irrational, $a+bx$ is irrational.

iii) For contradiction, assume $\sqrt{3}$ is rational.

i.e. $\sqrt{3} = \frac{p}{q}$ $\quad p, q \in \mathbb{Z}$, $q \neq 0$, $HCF(p,q) = 1$.

$\therefore 3 = \frac{p^2}{q^2}$

$p^2 = 3q^2$

$\therefore p^2$ is divisible by 3

$\therefore p$ is divisible by 3 (3 is prime)

i.e. $p = 3k$, $k \in \mathbb{Z}$.

$(3k)^2 = 3q^2$

$q^2 = 3k^2$

As above, q is divisible by 3.

This contradicts $HCF(p,q) = 1$.

$\therefore \sqrt{3}$ is irrational.

By ii) $5 + 7\sqrt{3}$ is irrational.

c) $x^2 - 2x + y^2 + z^2 + 4z + 4 = 0$

$$x^2 - 2x + (-1)^2 + y^2 + z^2 + 4z + (2)^2 = -4 + 1 + 4$$

$$(x-1)^2 + (y-0)^2 + (z+2)^2 = 1$$

$\therefore$ Centre $(1, 0, -2)$. Radius of 1.

Distance of centre from origin: $\sqrt{1^2 + 2^2}$

$= \sqrt{5} > 1$.

$\therefore$ The origin lies outside the sphere.

d)

$$L_1: \underset{\sim}{r}_1 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix}$$

$$L_2: \underset{\sim}{r}_2 = \begin{bmatrix} 3 \\ 8 \\ -2 \end{bmatrix} + \mu \begin{bmatrix} a-3 \\ -2-8 \\ 10--2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 8 \\ -2 \end{bmatrix} + \mu \begin{bmatrix} a-3 \\ -10 \\ 12 \end{bmatrix}.$$

$\therefore L_1 \parallel L_2$ if $\begin{bmatrix} a-3 \\ -10 \\ 12 \end{bmatrix} = \alpha \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix}$

So $-10 = 5\alpha$, then $a - 3 = 4\alpha$

$\alpha = -2$, $a - 3 = 4 \times -2$

$a = -5$

e) i) $\int_0^a f(x)\,dx$ Let $u = a - x$ $x = a - u$

$du = -dx$

$x = 0 \rightarrow u = a$

$x = a \rightarrow u = 0$

$= \int_a^0 f(a-u) - du$

$= \int_0^a f(a-u)\,du$

$= \int_0^a f(a-x)\,dx$

ii) $\int_0^1 x(1-x)^{50}\,dx$

$= \int_0^1 (1-x)\,x^{50}$ by i)

$= \int_0^1 x^{50} - x^{31}$

$= \left[\frac{x^{51}}{51} - \frac{x^{52}}{52}\right]_0^1$

$= \frac{1}{51} - \frac{1}{52}$

$= \frac{1}{2652}$

15.

a) i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{1}{2}e^{-4x}$

$\frac{1}{2}v^2 = \frac{1}{8}e^{-4x} + c_1$

At $x=0,\ t=0,\ v=\frac{1}{2}$.

$\therefore \frac{1}{2}\times\left(\frac{1}{2}\right)^2 = \frac{1}{8}e^0 + c_1$

$c_1 = 0$

$\frac{1}{2}v^2 = \frac{1}{8}e^{-4x}$

$v^2 = \frac{1}{4}e^{-4x}$

$v = \frac{1}{2}e^{-2x}$ as $v>0$ initially.

ii) $\frac{dx}{dt} = \frac{1}{2}e^{-2x}$

$\frac{dt}{dx} = 2e^{2x}$

$t = e^{2x} + c_2$

At $t=0,\ x=0$

$0 = e^0 + c_2$

$c_2 = -1$

$\therefore t = e^{2x} - 1$

$$e^{2x} = t+1$$
$$2x = \ln(t+1)$$
$$x = \frac{1}{2}\ln(t+1)$$

iii) As $t \to 0$

$$x = \ln(t+1)$$
$$\to \infty.$$
$$\therefore v = \frac{1}{2}e^{-2x}$$
$$\to \frac{1}{2} \times 0 = 0.$$

b)

$u = x^n \quad v' = \cos x$

$u' = nx^{n-1} \quad v = \sin x$

i) $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx$

$= \left[x^n \sin x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} n x^{n-1} \sin x \, dx$

$= \left(\frac{\pi}{2}\right)^n \times \sin \frac{\pi}{2} - 0 - n \int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx$

$u = x^{n-1} \quad v' = \sin x$

$u' = (n-1) x^{n-2} \quad v = -\cos x$

$= \left(\frac{\pi}{2}\right)^n - n \left(\left[-x^{n-1} \cos x \right]_0^{\frac{\pi}{2}} + \int (n-1) x^{n-2} \cos x \right) dx$

$= \left(\frac{\pi}{2}\right)^n - n \left(0 - 0 + (n-1) I_{n-2} \right)$

$= \left(\frac{\pi}{2}\right)^n - n(n-1) I_{n-2}$

ii) $I_4 = \left(\frac{\pi}{2}\right)^4 - 4 \times 3 \times I_2$

$= \left(\frac{\pi}{2}\right)^4 - 12 \left(\left(\frac{\pi}{2}\right)^2 - 2 \times 1 \times I_0 \right)$

$= \left(\frac{\pi}{2}\right)^4 - 12 \left(\frac{\pi}{2}\right)^2 + 24 \int_0^{\frac{\pi}{2}} \cos x \, dx$

$= \left(\frac{\pi}{2}\right)^4 - 12 \left(\frac{\pi}{2}\right)^2 + 24 \left(\sin \frac{\pi}{2} - \sin 0 \right)$

$= \left(\frac{\pi}{2}\right)^4 - 12 \left(\frac{\pi}{2}\right)^2 + 24$

e) i) $BE \perp AC$

$\therefore \lambda \underset{\sim}{b} \perp (\underset{\sim}{a} - \underset{\sim}{c})$

$\underset{\sim}{b} \perp (\underset{\sim}{a} - \underset{\sim}{c})$

$\therefore \underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{c}) = 0 \quad \text{—— (1)}$

Similarly $CF \perp AB$

$\mu \underset{\sim}{c} \perp (\underset{\sim}{b} - \underset{\sim}{a})$

$\underset{\sim}{c} \perp (\underset{\sim}{b} - \underset{\sim}{a})$

$\underset{\sim}{c} \cdot (\underset{\sim}{b} - \underset{\sim}{a}) = 0 \quad \text{—— (2)}$

ii) From (1) $\underset{\sim}{b} \cdot \underset{\sim}{c} - \underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

$\underset{\sim}{b} \cdot \underset{\sim}{c} = \underset{\sim}{a} \cdot \underset{\sim}{b}$

From (2) $\underset{\sim}{c} \cdot \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{c} = 0$

$\underset{\sim}{b} \cdot \underset{\sim}{c} = \underset{\sim}{a} \cdot \underset{\sim}{c}$

$\therefore \underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{a} \cdot \underset{\sim}{c}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{c} = 0$

$\underset{\sim}{a} \cdot (\underset{\sim}{b} - \underset{\sim}{c}) = 0$

$\therefore \overrightarrow{AO} \perp \overrightarrow{BC}$

$\overrightarrow{AD} \perp \overrightarrow{BC}$

iii) Altitudes of a triangle are concurrent.

16.

a)

i) $LHS = (1+i\tan\theta)^n + (1-i\tan\theta)^n$

$$= \left(1+\frac{i\sin\theta}{\cos\theta}\right)^n + \left(1-\frac{i\sin\theta}{\cos\theta}\right)^n$$

$$= \left(\frac{\cos\theta+i\sin\theta}{\cos\theta}\right)^n + \left(\frac{\cos\theta-i\sin\theta}{\cos\theta}\right)^n$$

$$= \frac{(\cos\theta+i\sin\theta)^n + (\cos(-\theta)+i\sin(-\theta))^n}{\cos^n\theta}$$

$$= \frac{\cos n\theta + i\cancel{\sin n\theta} + \cos(-n\theta) + i\cancel{\sin(-n\theta)}}{\cos^n\theta}$$

$$= \frac{2\cos(n\theta)}{\cos^n\theta}$$

ii) If $(1+z)^4 + (1-z)^4 = 0$

Let $z = i\tan\theta$, by i):

$$\frac{2\cos 4\theta}{\cos^4\theta} = 0$$

$$\cos 4\theta = 0$$

$$4\theta = \frac{\pi}{2}, \frac{3\pi}{2}, -\frac{\pi}{2}, -\frac{3\pi}{2}$$ Need 4 solutions.

$$\theta = \pm\frac{\pi}{8}, \pm\frac{3\pi}{8}$$

$$\therefore z = \pm i\tan\frac{\pi}{8}, \pm i\tan\frac{3\pi}{8}$$

b) i) $a=6,\ b=10,\ c=15.$

ii) Test $n=2$.

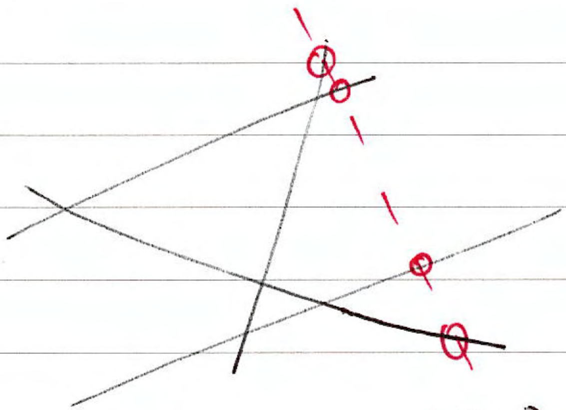
$$X_1 = \frac{2(2-1)}{2}$$
$= 1$, as on the table.

$\therefore$ Statement true for $n=1$.

Assume true for $n=k$, i.e. $X_k = \frac{k(k-1)}{2}$.

Hence show true for $n=k+1$, RTP $X_{k+1} = \frac{(k+1)k}{2}$

Now, the $(k+1)^{th}$ line will maximally intersect k lines:



Intersects with every available line.

$\therefore X_{k+1} = X_k + k = \frac{k(k-1)}{2} + k$, by assumption

$$= \frac{k^2 - k + 2k}{2}$$

$$= \frac{(k+1)k}{2},$$

as required.

$\therefore$ If the statement is true for $n=k$, it is true for $n=k+1$. As it is true for $n=1$, it is true for $n=1+1=2, 3, \ldots$.
Hence by mathematical induction, it is true for all positive integers.

c) i) $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 \geq 0$

$$x - 2 + \frac{1}{x} \geq 0$$

$$x + \frac{1}{x} \geq 2.$$

ii) $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

$$= 1+1+1+\frac{a}{b}+\frac{b}{a}+\frac{a}{c}+\frac{c}{a}+\frac{b}{c}+\frac{c}{b}$$

$$= 3+\frac{a}{b}+\frac{1}{\left(\frac{a}{b}\right)}+\frac{a}{c}+\frac{1}{\left(\frac{a}{c}\right)}+\frac{b}{c}+\frac{1}{\left(\frac{b}{c}\right)}$$

$\geq 3+2+2+2$ by i)

$= 9.$

iii) In ii), let $a=a+b$, $b=a+c$, $c=b+c$:

$$\left((a+b)+(a+c)+(b+c)\right)\left(\frac{1}{a+b}+\frac{1}{a+c}+\frac{1}{b+c}\right) \geq 9$$

$$2(a+b+c)\left(\frac{1}{a+b}+\frac{1}{a+c}+\frac{1}{b+c}\right) \geq 9$$

$$\left(\frac{a}{a+b}+\frac{b}{a+b}\right)+\left(\frac{a}{a+c}+\frac{c}{a+c}\right)+\left(\frac{b}{b+c}+\frac{c}{b+c}\right)$$

$$+\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b} \geq \frac{9}{2}$$

$$1+1+1+\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b} \geq \frac{9}{2}$$

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b} \geq \frac{3}{2}$$